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Motion of a classical particle with spin: II. Calculation of Dirac brackets

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Abstract. We continue the canonical formalism of paper I for the motion of a free particle. We establish by the use of a computer the relevant matrices necessary for the computation of all Dirac brackets. An algorithm is described which allots numerical values to the canonical variables satisfying the constraints, and these values are used in the construction of the matrices. Algebraic expressions for Dirac brackets are predicted from a first set of numerical data, and these predicted expressions are confirmed by at least two further sets of data. Reduction to a sufficient set of variables is attempted and the reduced set expressed as a Lie algebra.

1. Introduction

In paper I (Ellis 1982) the classical covariant equations of motion of a free spinning particle with moments of inertia I_{ij} were derived canonically from a degenerate Lagrangian. The degeneracy of the Lagrangian arose from the (necessary) introduction of Lagrange multiplier terms. Using Dirac's method of canonical multiplier functions (Dirac 1964), modified to take account of the existence of first-order Lagrange equations (see e.g. Shanmugadhasan 1973), we derived the complete canonical formalism, with constraints, for these well known equations generalised to include asymmetry:

$$\dot{p}^{\mu}=0, \qquad \dot{s}^{\mu\nu}+2p^{[\mu}\dot{x}^{\nu]}=0, \qquad s^{\mu\nu}\dot{x}_{\nu}=0.$$

The spin tensor $s^{\mu\nu}$ had the structure given by the equations

$$s^{\mu\nu}=-c\varepsilon^{\mu\nu\lambda\tau}s_{\lambda}\dot{x}_{\tau}, \qquad s_{\lambda}=s_{i}u_{i\lambda}, \qquad s_{i}=I_{ij}\omega_{j}.$$

This canonical method for a degenerate Lagrangian, modified to take account of first-order Lagrange equations, is not especially well known and its application to the classical equations has not previously been attempted‡.

In this paper we carry out the detailed calculation of Dirac brackets (DBs). Paper I has already dealt with the problem in general, and has explained the calculation of the complete set of canonical constraints and the Hamilton equations for the model. The canonical constraints (called the set of subsidiary conditions on the canonical variables) are weak equations, and in I we derived the matrix of mutual Poisson

† This work was carried out in part while on leave of absence between October 1979 and September 1980.
‡ Note added in final draft: the implications of a recent letter by Schafir (1982) are discussed in a footnote to § 2.

brackets (PBs) of all these constraints and determined those that were second class†. The second-class constraints are identities with respect to the new brackets: it does not matter whether these constraints are used before or after the calculation of the new brackets, unlike the situation with regard to PBs.

In I we found that the only first-class constraints were the zero momenta conjugate to the Lagrange multipliers whose values are not required for the derivation of the equations of motion. Thus these first-class variables and their conjugates drop out of consideration and we find the DBs for all other variables according to the definition of the bracket. (Note that in I iterative methods were not used in the calculation‡.) This calculation of DBs appears to be straightforward, but complications arise from the actual task of computation, especially the calculation of the inverse of the large antisymmetrical submatrix of mutual PBs of the second-class constraints (I, table 1), required in the definition of the DB. Since we are dealing with a relativistic model, a method for calculating the algebraic form of this matrix and of the DBs themselves must be found that does not destroy the manifest covariance of the model. Even if this calculation is possible, the task of finding certain DBs of the canonical variables may be too difficult, and consequently we may not be able to compute other DBs involving products of canonical variables by using the product rule for DBs; it is, therefore, advantageous to include some physical components among the variables whose DBs we wish to compute, in order that the DBs of these more difficult products may be computed.

In the following calculation we have made use of a computer. This use is a novel one which, so far as we are aware, has not before been considered as a possible alternative to algebraic calculation. We use a computer to find both the pattern for the large matrix inverse and to predict algebraic values for DBs. Because of the constraint equations, we do not use completely arbitrary numerical values of the canonical variables, but values that are consistent with the constraints. Thus we make the computer carry out the right large-scale matrix operations numerically subject to these constraints (the process can be made automatic by the use of the random number generator) and we examine the output for equalities and relationships between the matrix elements. In this way we predict from the numerical values both the large matrix inverse and algebraic values for most of the DB relations. (The inverse has been confirmed conclusively by straightforward algebra, and the DB relations have been tested by comparison with further sets of numerical data§.) We thus sidestep the task of algebraic calculation and are able to reduce drastically the size of the algebraic problem.

The numbers of variables used in these calculations are as follows: number of coordinates and velocities $(2n) = 52$; number of subsidiary conditions on the canonical variables $(r_3) = 30$; number of second-class constraints $(r_4) = 24$. The number of independent combinations of canonical variables, sufficient for the description of the

† First-class constraints have no effect on the equations of motion and correspond to rows and columns of zeros in the PB matrix. The submatrix of mutual PBs of the second-class constraints is of even dimensionality and non-singular, enabling the calculation of all the canonical multipliers corresponding to the second-class constraints to be made. The inverse of this submatrix is used in the definition of the DB (see below).

‡ We used non-iterative methods because of the existence of first-order Lagrange equations and because not all the velocities may be solved simply in terms of the coordinates and momenta.

§ This is, of course, not a mathematically conclusive test of the DB relations, but their agreement with further sets of data to an accuracy of about 10^{-8} makes it extremely unlikely that the predicted expressions are incorrect.

model, is $2n_1$ where $n_1 =$ (number of degrees of freedom) − (number of first-class constraints). This number of variables is obtained by the elimination of certain variables using the second-class constraints provided we use only the new brackets, since these constraints are identities with respect to DBs. The number of degrees of freedom of the model is $n - \frac{1}{2}r_4$ (this number includes the first-class variables and their conjugates).

In § 2, we give the definition of the DB that involves the weak inverse of the submatrix of mutual PBs of the second-class constraints, as opposed to the strong inverse. We also give some properties of the DB. In § 3 we calculate by numerical methods the required weak inverse for the problem and compute the DBs of most of the canonical variables and combinations of variables that represent physical components. Also in § 3 we attempt the reduction of the system to a sufficient set of variables.

2. Definition and properties of the Dirac bracket

In I we used a modified version of the multiplier formalism in which all of the constraints arose on an equal footing in the total Hamiltonian, and in which DBs were not calculated iteratively. The form of this theory required all the constraints to be known in advance†. Below we give the definition of the DB in which the *weak inverse* of the matrix of mutual PBs of the second-class constraints, as opposed to the strong inverse, is used. (The strong inverse is the inverse obtained without the use of the constraints (cf Hanson and Regge 1974).)

Let ϕ_A $(A = 1, 2, \ldots, r_4)$ denote the second-class constraints taken in any order, and multiplied by renormalising constants if necessary. Let $H = H_0 + u_A\phi_A$ represent the total Hamiltonian, where the canonical multipliers u_A are obtained from the consistency conditions as described in I. (These u_A are appropriate multiples of the

† A recent letter by Schafir (1982) has drawn attention to the controversy concerning the necessity (or otherwise) for including in the total Hamiltonian other constraints besides the primary constraints. The arguments in the present case are difficult to apply because of the complicated nature of the equations; but it is easy to see that in the spinless case ($I_{ij} = 0$), the usual iterative Dirac theory correctly gives the secondary constraint $p^\mu p_\mu \approx m_0^2 c^4$, starting from the single primary constraint $\Pi_0 \approx 0$.

In the non-spinless case ($I_{ij} \neq 0$), which we primarily had in mind in I, we stated that it was 'quite clear that the constraints (3.4) cannot be determined from (3.3) (the primary constraints) using the current theory of multipliers'. This statement was unfortunately too strong. We ought to have said 'it is not obvious that the constraints (3.4) can be determined…', for in the case $I_{ij} \neq 0$ it is difficult (though perhaps not theoretically impossible) to express the Hamiltonian $H_0 = -p^\mu \dot{x}_\mu - \pi_i^\mu \dot{u}_{i\mu} - \ldots - L$ in terms of coordinates and momenta without resorting to the 'secondary' constraints (3.4), which ought to be derived from Dirac's theory. Because of this difficulty and because of the known work of Shanmugadhasan (1973), we took *all* the constraints on an equal footing, simplified H_0 using them all, and allowed canonical multipliers for all the constraints in the total Hamiltonian. This method (which, of course, is perfectly legitimate provided all the constraints are known) amounts to the final stage of Dirac's iterative procedure, where all the constraints have been found, except that since a weakly equivalent expression for H_0 has been used in the total Hamiltonian, the 'secondary' constraints may now have *non-zero* multipliers attached to them. We do not assert that this is the only way to do things; we merely state that the method of Shanmugadhasan appears to avoid the technical difficulties. Indeed, if, as now seems likely, the first-order Lagrange equations are nearly always theoretically derivable as secondary constraints from the usual iterative theory, starting from the primary constraints, his method seems a reasonable one to adopt. (In I, the Hamiltonian equations for the physical variables were checked independently against the Lagrangian ones, and all these equations were found to be correct.)

μ's, μ''s and ν's already found in I. The r_3-r_4 arbitrary multipliers multiplying the first-class constraints have been set to zero.)

The canonical equation of motion (I, equation (4.4)) for any function g of the canonical variables and τ is

$$dg/d\tau \approx \partial g/\partial\tau - \{g, H_0\} - \{g, \phi_A\}u_A, \tag{2.1}$$

since the PBs $\{g, u_A\}$ are multiplied by the ϕ's which vanish weakly. The r_4 consistency conditions are the conditions obtained from (2.1) by setting g equal to the r_4 ϕ's in turn:

$$0 = \dot{\phi}_B \approx \partial\phi_B/\partial\tau - \{\phi_B, H_0\} - \{\phi_B, \phi_C\}u_C. \tag{2.2}$$

An expression representing the definition of the DB arises from (2.1) by eliminating the multipliers u_C in terms of PBs from (2.2), as follows. Denote by C_{AB} the PB $\{\phi_A, \phi_B\}$, which is one element of the r_4-dimensional submatrix $\boldsymbol{C}$ of the r_3-dimensional matrix of mutual PBs of all the constraints calculated in I. This submatrix refers to the second-class constraints only and is necessarily non-singular. Denote by C_{AB}^{-1} the elements of the (weak) inverse of $\boldsymbol{C}$ determined, using the second-class constraints if required, from

$$C_{AB}^{-1}C_{BC} \approx \delta_{AC} \tag{2.3}$$

(as usual, summation is implied by repeated capital letters for the range $1, \ldots, r_4$). Contracting (2.2) with C_{AB}^{-1}, we obtain

$$u_A \approx C_{AB}^{-1}(\partial\phi_B/\partial\tau - \{\phi_B, H_0\}),$$

and with these values the canonical equation of motion (2.1) for the function g becomes†

$$dg/d\tau \approx \partial g/\partial\tau - \{g, \phi_A\}C_{AB}^{-1}\,\partial\phi_B/\partial\tau - (\{g, H_0\} - \{g, \phi_A\}C_{AB}^{-1}\{\phi_B, H_0\}). \tag{2.4}$$

The term in parentheses represents the Dirac bracket (or the 'modified' or 'restricted' PB) of g and H_0. The DB of two functions ξ, η of the canonical variables is thus defined in terms of PBs and the weak inverse of $\boldsymbol{C}$ by the equation

$$\{\xi, \eta\}^* \stackrel{\text{def}}{=} \{\xi, \eta\} - \{\xi, \phi_A\}C_{AB}^{-1}\{\phi_B, \eta\}. \tag{2.5}$$

The use of the second-class constraints in the elements of the inverse of $\boldsymbol{C}$ has no effect on either the DB or the equations of motion (2.4) other than that which is permitted by the adjoined subsidiary conditions. In the present calculation the second-class constraints do not explicitly depend on τ, and the second term of (2.4) vanishes. In this case the equation of motion for g is

$$dg/d\tau = \partial g/\partial\tau - \{g, H_0\}^*. \tag{2.6}$$

The DB (2.5) is defined in such a way that all the DBs of the second-class constraints ϕ_A with themselves and with any function η of the canonical variables vanish weakly, i.e.

$$\{\phi_A, \phi_B\}^* \approx 0, \qquad \{\phi_A, \eta\}^* \approx 0.$$

† This equation has been given by Shanmugadhasan (1973, equation (45)), where C_{AB}^{-1} denotes the strong inverse, and by Sudarshan and Mukunda (1974, p 104, equation (96)), where the weak inverse is used; Sudarshan and Mukunda omit τ dependence but include terms arising from arbitrary multiples of the first-class constraints, which may without loss of generality be taken as zero, as already mentioned in I.

('Weakly' here means by the use of the second-class constraints only, since their use is allowed in (2.3).) These relations show that the second-class constraints are *identities* with respect to the new brackets, and that it is immaterial whether the second-class constraints are used before or after working out the new brackets†.

The DB (2.5) has all the properties of a Lie product binary operation (antisymmetry, linearity, Jacobi identity) together with a product rule similar to that found for PBs. There is also a 'function of a function' rule, as for PBs, where the elements in the bracket depend on several independent functions of the canonical variables. Thus the DB is the natural generalisation of the PB to be adopted in the Dirac correspondence when there exist independent constraints between the phase-space variables.

3. The calculation of Dirac brackets

3.1. The matrix of mutual Poisson brackets of the constraints

In the canonical formalism the $2n$ canonical variables are the 26 coordinates x^μ, u_i^μ, m_0, λ_{0i}, λ_{ij} ($\equiv \lambda_{ji}$) and the 26 momenta p_μ, $\pi_{i\mu}$, Π_0, Π_{0i}, Π_{ij} ($\equiv \Pi_{ji}$). The PB is defined with respect to all 26 pairs of canonical variables. In I we showed that the r_3 subsidiary conditions on the canonical variables are the following 30 constraints (represented by weakly vanishing functions):

$$\phi_0 \overset{\text{def}}{=} \Pi_0 \approx 0, \qquad \phi_{0i} \overset{\text{def}}{=} \Pi_{0i} \approx 0, \qquad \phi_{ij} \overset{\text{def}}{=} \Pi_{ij} \approx 0,$$

$$\phi'_0 \overset{\text{def}}{=} m_0^2 c^4 (v^\mu v_\mu - 1) \approx 0, \qquad \phi'_{0i} \overset{\text{def}}{=} m_0 c^2 v^\mu u_{i\mu} \approx 0,$$

$$\phi'_{ij} \overset{\text{def}}{=} u_i^\mu u_{j\mu} + \delta_{ij} \approx 0, \qquad \chi_{0i} \overset{\text{def}}{=} m_0 c^2 v^\mu \pi_{i\mu} \approx 0,$$

$$\chi_{ij} \overset{\text{def}}{=} \pi_i^\mu u_{j\mu} + \pi_j^\mu u_{i\mu} \approx 0, \qquad \chi_0 \overset{\text{def}}{=} \lambda_{0i} s_i \approx 0. \tag{3.1}$$

The notation v^μ, s_i represents the four-velocity and the 'internal components' of spin defined as combinations of canonical variables:

$$v^\mu \overset{\text{def}}{=} (p^\mu + \lambda^\mu)/m_0 c^2 \qquad (\lambda^\mu \overset{\text{def}}{=} \lambda_{0i} u_i^\mu), \tag{3.2}$$

$$s_i \overset{\text{def}}{=} c^{-1} \varepsilon_{ijk} u_j^\sigma \pi_{k\sigma}. \tag{3.3a}$$

In addition, the following notation is used for the spin, orbital and total angular momentum tensors:

$$s^{\mu\nu} \overset{\text{def}}{=} -2u_i^{[\mu} \pi_i^{\nu]}, \qquad m^{\mu\nu} \overset{\text{def}}{=} -2x^{[\mu} p^{\nu]}, \qquad j^{\mu\nu} \overset{\text{def}}{=} m^{\mu\nu} + s^{\mu\nu}. \tag{3.3b}$$

A canonical spin four-vector is also defined:

$$s^\mu \overset{\text{def}}{=} s_i u_i^\mu. \tag{3.4}$$

† We have replaced the symbol '$\approx$' in (2.6) by '$=$' in view of this fact.

The $r_3 - r_4$ first-class constraints are the six constraints $\phi_{ij} \approx 0$, the remaining constraints being second class. The simplified Hamiltonian (I, equation (4.2)) is

$$H_0 = -m_0 c^2 + \tfrac{1}{2} I^{-1}_{pq} s_p s_q, \tag{3.5}$$

and the first-class Hamiltonian H, for which the $n-(r_3-r_4)$ pairs of Hamilton's equations lead to the correct equations of motion (by the use of the constraints), has also been calculated.

The mutual PBs of all 30 canonical constraints (3.1) are derived and listed in I (table 1). The first-class nature of the functions ϕ_{ij} is exhibited by the columns and rows of zeros adjacent to these functions in the table. The omission of these columns and rows from the table forms the 24-dimensional non-singular submatrix $\boldsymbol{C}$, in which the rows and columns are those that correspond to the second-class constraints. The ordering and normalisation of these 24 second-class constraints are taken, for the definition of this matrix, to be ϕ_A ($A = 1, \ldots, 24$): $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_8$, where the $\boldsymbol{\phi}$'s represent the following consecutive groups of three constraints: $c^{-2}\phi_0$, $\frac{1}{2}\phi'_0$, $c\chi_0$; $\phi_{01}, \ldots$; $\phi'_{01}, \ldots$; $\chi_{01}, \ldots$; $\phi'_{11}, \ldots$; $\phi'_{23}, \ldots$; $\chi_{11}, \ldots$; $2\chi_{23}, \ldots$. This order is convenient for finding the inverse. From the table of mutual PBs of the constraints given in I, and with this ordering and normalisation, the matrix $\boldsymbol{C}$, partitioned into 16 square sub-matrices of dimension six, is

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{\beta}^{\mathrm{T}} & \boldsymbol{\gamma} & \boldsymbol{0} & \boldsymbol{\Lambda} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -4\boldsymbol{I} \\ \boldsymbol{0} & -\boldsymbol{\Lambda}^{\mathrm{T}} & 4\boldsymbol{I} & \boldsymbol{S} \end{pmatrix}, \tag{3.6}$$

where

$$\boldsymbol{\alpha} = \begin{pmatrix} m_0 c^2 \boldsymbol{\varepsilon} & \boldsymbol{\tau} \\ -\boldsymbol{\tau}^{\mathrm{T}} & \boldsymbol{0} \end{pmatrix}, \qquad \boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{0} & m_0^2 c^4 \boldsymbol{\xi} \\ \boldsymbol{I} & \frac{1}{2}\boldsymbol{\sigma} \end{pmatrix}, \qquad \boldsymbol{\gamma} = m_0^2 c^4 \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0} \end{pmatrix},$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{\Lambda}_{11} & \boldsymbol{\Lambda}_{12} \\ -\frac{1}{2}\boldsymbol{\sigma}\boldsymbol{\Lambda}_{11} & -\frac{1}{2}\boldsymbol{\sigma}\boldsymbol{\Lambda}_{12} \end{pmatrix}, \qquad \boldsymbol{S} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{S}_{12} \\ -\boldsymbol{S}_{12}^{\mathrm{T}} & -4\boldsymbol{\sigma} \end{pmatrix}.$$

The notation used in the right-hand sides of these last five equations represents square matrices of dimension three, and $\boldsymbol{\varepsilon}$, $\boldsymbol{\tau}$, $\boldsymbol{\xi}$, $\boldsymbol{\Lambda}_{11}$, $\boldsymbol{\sigma}$, $\boldsymbol{\Lambda}_{12}$ and $\boldsymbol{S}_{12}$ are defined thus: $\boldsymbol{\varepsilon}$ has 1 and -1 in its (1, 2) and (2, 1) positions, $\boldsymbol{\tau}$ has components cs_i in its third row, $\boldsymbol{\xi}$ has components λ_{0i} in its second row, $\boldsymbol{\Lambda}_{11}$ has components $-2\lambda_{0i}$ in its main diagonal, and

$$\boldsymbol{\sigma} = c\begin{pmatrix} \cdot & -s_3 & s_2 \\ s_3 & \cdot & -s_1 \\ -s_2 & s_1 & \cdot \end{pmatrix}, \qquad \boldsymbol{\Lambda}_{12} = -2\begin{pmatrix} \cdot & \lambda_{03} & \lambda_{02} \\ \lambda_{03} & \cdot & \lambda_{01} \\ \lambda_{02} & \lambda_{01} & \cdot \end{pmatrix},$$

$$\boldsymbol{S}_{12} = 4c\begin{pmatrix} \cdot & s_2 & -s_3 \\ -s_1 & \cdot & s_3 \\ s_1 & -s_2 & \cdot \end{pmatrix}. \tag{3.7}$$

(The antisymmetry of (3.6) can be shown from the antisymmetry of $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{S}$.)

3.2. Computation of $\boldsymbol{C}^{-1}$

The calculation of the algebraic form of the inverse of the 24-dimensional matrix (3.6), simplified by using the constraints, is not an easy task to perform by conventional means. We attempt this calculation by numerical means. An algorithm is used for allotting values to all the canonical variables (excluding the first-class variables and their conjugates) so that these values simultaneously satisfy all 24 second-class constraint equations†. With these values a computer is then made to construct the matrix $\boldsymbol{C}$ and invert it. From the numerical values obtained we predict the precise algebraic form of the weak inverse, and this is checked algebraically. In the algorithm described below, the four-dimensional metric matrix diag(1, -1, -1, -1) is denoted by $\boldsymbol{G}$.

The moving tetrad $(\dot{x}, u_i^\mu)$ is represented by a real four-dimensional numerical Lorentz matrix $\boldsymbol{U}$, created from a rotation $\phi\hat{\boldsymbol{n}}$ followed by a special Lorentz transformation with velocity $\boldsymbol{v} = -c(\tanh\chi)\hat{\boldsymbol{v}}$ in which $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{v}}$ are freely chosen, $-\pi \leqslant \phi < \pi$ and χ is chosen sufficiently small to prevent the problem from becoming ill conditioned. Six arbitrary parameters are involved in $\boldsymbol{U}$. The arithmetic is such that $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{U}$ and $\boldsymbol{G}$ differ by quantities not exceeding 10^{-9}. When this is achieved, the sixth equation of (3.1) is verified for these values of u_i^μ to within the desired accuracy. Four parameters formed by the row matrix of four elements $\boldsymbol{L} = (m_0c^2, \lambda_{01}, \lambda_{02}, \lambda_{03})$ are chosen, where $m_0^2c^4 > |\boldsymbol{\lambda}|^2$, $0 < m_0c^2 < 3$ and $-1 < \lambda_{0i} < 1$. The row matrix $\boldsymbol{P} = \boldsymbol{L}\boldsymbol{U}^{-1}\boldsymbol{G}$ represents p^μ, and this is computed. This enables an arithmetical check to be performed on the first column of $\boldsymbol{U}$ (representing $\dot{x}^\mu$) and on the fourth and fifth constraints of (3.1). There remain two parameters that have not yet been fixed. These are taken to be the first two components of $\boldsymbol{s} = (cs_1, cs_2, cs_3)$, chosen arbitrarily between -1 and 1. The third component is computed from the ninth constraint of (3.1). The computation of the π's is made from the equation $\boldsymbol{\pi}^\mu = \frac{1}{2}\boldsymbol{s} \wedge \boldsymbol{u}^\mu$ (I, appendix 2). By using these components, we construct the square matrices

$$\boldsymbol{V} = (p^\mu, \pi_1^\mu, \pi_2^\mu, \pi_3^\mu), \qquad \boldsymbol{W} = \left(\boldsymbol{L}^{\mathrm{T}} \left| \begin{matrix} \boldsymbol{0} \\ -\frac{1}{2}\boldsymbol{\sigma} \end{matrix} \right.\right).$$

Provided the elements of $\boldsymbol{W}$ do not differ from those of $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{V}$ by quantities exceeding 10^{-9} this ensures that the remaining seventh and eighth constraints of (3.1) are satisfied to within the desired accuracy.

This algorithm uses 12 arbitrary parameters‡ and thus produces the values of all the canonical variables (excluding the four x's, which are not used, and the four Π's, which are already zero from the constraints (3.1)). Numerical values of the canonical variables are obtained from the matrices $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$. The matrix $\boldsymbol{W}$ is used directly in computing the elements of (3.6), whose inverse is then computed. We predict a general algebraic expression for the inverse, and by partitioning this matrix into submatrices of dimension six we find all the elements of the weak inverse:

$$\boldsymbol{C}^{-1} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{\Delta} & \boldsymbol{0} \\ -\boldsymbol{B}^{\mathrm{T}} & \boldsymbol{\Gamma} & \boldsymbol{E} & \boldsymbol{0} \\ -\boldsymbol{\Delta}^{\mathrm{T}} & -\boldsymbol{E}^{\mathrm{T}} & \boldsymbol{Z} & \frac{1}{4}\boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} & -\frac{1}{4}\boldsymbol{I} & \boldsymbol{0} \end{pmatrix}. \tag{3.8}$$

† The matrix (3.6) contains only some of the canonical variables, and we require an algorithm for creating arbitrary values of these subject only to χ_0 since this second-class constraint is the only one restricting the choice of these variables. The more general algorithm is described which allots arbitrary values to all the canonical variables subject to all the second-class constraints, since this algorithm is required later in the calculation of DBs.

‡ These arbitrary parameters and the x's correctly number $2n_1 = 16$.

The capital letters represent square matrices of dimension six:

$$\boldsymbol{A} = -\frac{1}{|\boldsymbol{s}|^2}\begin{pmatrix} |\boldsymbol{s}|^2\boldsymbol{\varepsilon}/m_0c^2 & \boldsymbol{\kappa} \\ -\boldsymbol{\kappa}^{\mathrm{T}} & m_0^2c^4\boldsymbol{\sigma} \end{pmatrix}, \qquad \boldsymbol{B} = -\frac{1}{|\boldsymbol{s}|^2}\begin{pmatrix} -\frac{1}{2}|\boldsymbol{s}|^2\boldsymbol{\eta}/m_0c^2 & \boldsymbol{\kappa}/m_0^2c^4 \\ -\frac{1}{2}\boldsymbol{\sigma}^2 & \boldsymbol{\sigma} \end{pmatrix},$$

$$\boldsymbol{\Gamma} = -\frac{1}{m_0^2c^4|\boldsymbol{s}|^2}\begin{pmatrix} \frac{1}{4}|\boldsymbol{s}|^2\boldsymbol{\sigma} & \frac{1}{2}\boldsymbol{\sigma}^2+|\boldsymbol{s}|^2\boldsymbol{I} \\ -\frac{1}{2}\boldsymbol{\sigma}^2-|\boldsymbol{s}|^2\boldsymbol{I} & \boldsymbol{\sigma} \end{pmatrix},$$

$$\boldsymbol{\Delta} = \tfrac{1}{4}\boldsymbol{B}\boldsymbol{\Lambda} = \frac{1}{4|\boldsymbol{s}|^2}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{\sigma}^2\boldsymbol{\Lambda}_{11} & \boldsymbol{\sigma}^2\boldsymbol{\Lambda}_{12} \end{pmatrix},$$

$$\boldsymbol{E} = \tfrac{1}{4}\boldsymbol{\Gamma}\boldsymbol{\Lambda} = \frac{1}{4m_0^2c^4|\boldsymbol{s}|^2}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{\sigma}^2\boldsymbol{\Lambda}_{11}+|\boldsymbol{s}|^2\boldsymbol{\Lambda}_{11} & \boldsymbol{\sigma}^2\boldsymbol{\Lambda}_{12}+|\boldsymbol{s}|^2\boldsymbol{\Lambda}_{12} \end{pmatrix},$$

$$\boldsymbol{Z} = \tfrac{1}{16}(\boldsymbol{S}+\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{\Lambda}) = \tfrac{1}{16}\begin{pmatrix} \mathbf{0} & \boldsymbol{S}_{12} \\ -\boldsymbol{S}_{12}^{\mathrm{T}} & -4\boldsymbol{\sigma} \end{pmatrix}.$$

The notation used in the right-hand sides of these six equations represents square matrices of dimension three, and, in addition to the matrices listed previously, $\boldsymbol{\kappa}$ has $m_0c^2\boldsymbol{\lambda}\wedge\boldsymbol{s}$ and $\boldsymbol{s}$ for its first and third rows, $\boldsymbol{\eta}$ has $\boldsymbol{\lambda}$ for its first row ($\boldsymbol{\lambda}$ denotes the triple $(\lambda_{01}, \lambda_{02}, \lambda_{03})$). The matrix (3.8) has been verified algebraically.

3.3. *The computation of* DBs

For any two canonical variables (or functions of canonical variables) ξ, η, the DB is defined by

$$\{\xi, \eta\}^* = \{\xi, \eta\} + \boldsymbol{\xi}\boldsymbol{C}^{-1}\boldsymbol{\eta}^{\mathrm{T}} \tag{3.9}$$

from (2.5), where $\boldsymbol{C}^{-1}$ is the weak inverse (3.8), and $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ denote 24-element row matrices whose elements are the PBs of ξ and η with the 24 second-class constraints. The variables ξ and η are taken in turn to be the canonical variables x^α, p^α, u_i^α, π_i^α, m_0c^2, $c^{-2}\Pi_0$, λ_{0i}, Π_{0i} and the combinations (3.3), which represent physical components. These variables are the ones whose PBs with the second-class constraints are required. The PBs of the combinations (3.3) with the second-class constraints are obtained from the other PBs by using the product rule (for PBs). The calculation of DBs is not especially complicated by having an extra number of variables and PBs since the new variables and PBs merely represent further data in the computer calculation.

We omit the list of PBs of the canonical variables and special combinations with the second-class constraints, and give only those non-zero PBs of the x's and p's that enable us to calculate the corresponding DBs algebraically:

$$\{x^\alpha, \boldsymbol{\phi}_1\} = (0, p^\alpha + \boldsymbol{\lambda}\cdot\boldsymbol{u}^\alpha, 0), \qquad \{x^\alpha, \boldsymbol{\phi}_3\} = \boldsymbol{u}^\alpha, \qquad \{x^\alpha, \boldsymbol{\phi}_4\} = \boldsymbol{\pi}^\alpha.$$

Each DB (3.9) is the sum of a PB and a scalar computed by matrix multiplication. This scalar is the sum of $24+24^2=600$ products of elements, and it may be simplified by using any of the second-class constraints. Some of these products are zero, but the algebraic task is formidable. The computer calculates the 61-dimensional matrix of scalars $\boldsymbol{\xi}\boldsymbol{C}^{-1}\boldsymbol{\eta}^{\mathrm{T}}$ (some of these are redundant; many are zero), and the algebraic results predicted from the numerical ones are listed in table 1 below.

Since the mutual PBs of the x's and p's vanish, their DBs likewise vanish by table 1, and this is confirmed by the algebraic calculations referred to. All other results in

Table 1. The difference $\{\xi, \eta\}^* - \{\xi, \eta\}$ as computed from numerical values of the scalar $\boldsymbol{\xi} \boldsymbol{C}^{-1} \boldsymbol{\eta}^{\mathrm{T}}$.

$\eta \rightarrow$ $\xi \downarrow$	x^μ p^μ $m^{\mu\nu}$ $s^{\mu\nu}$ cs_i $j^{\mu\nu}$	u_i^μ	π_i^μ	m_0c^2	$c^{-2}\Pi_0$	λ_{0j}	Π_{0j}
x^α		$-\dfrac{s^\alpha s_i v^\mu}{m_0c^2 s_k s_k}$	0	v^α	0	$\dfrac{2}{cs_k s_k}\varepsilon_{ipq} s_p \pi_q^\alpha$	0
p^α		0	0	0	0	0	0
$m^{\alpha\beta}$	(Block of 36 zeros)	$-\dfrac{2p^{[\alpha}s^{\beta]}s_i v^\mu}{m_0c^2 s_k s_k}$	0	$2p^{[\alpha}v^{\beta]}$	0	$\dfrac{4}{cs_k s_k}\varepsilon_{ipq} s_p p^{[\alpha}\pi_q^{\beta]}$	0
$s^{\alpha\beta}$		$\dfrac{2p^{[\alpha}s^{\beta]}s_i v^\mu}{m_0c^2 s_k s_k}$	0	$-2p^{[\alpha}v^{\beta]}$	0	$\dfrac{-4}{cs_k s_k}\varepsilon_{ipq} s_p p^{[\alpha}\pi_q^{\beta]}$	0
cs_i		0	0	0	0	$\varepsilon_{ijk}\lambda_{0k}$	0
$j^{\alpha\beta}$		0	0	0	0	0	0
u_i^α		—	—	$-m_0c^2\nu_{0i}v^\alpha$	0	—	0
π_i^α			—	$-\frac{1}{2}\lambda_{0i}v^\alpha$	0	—	0
m_0c^2				0	-1	$m_0^2c^4\nu_{0j}$	0
$c^{-2}\Pi_0$					0	0	0
λ_{0i}						$\dfrac{m_0^2c^3}{s_k s_k}\varepsilon_{ijk} s_k$	$-\delta_{ij}$
Π_{0i}							0

table 1 are predicted from the numerical ones†. This table represents the quantities that must be added to PBs in order to convert PBs into DBs. Some expressions have not been found (represented by dashes in the table). A complete block of 36 zeros has been found for the variables x^α, p^α and the special combinations (3.3). Three blocks are suppressed because of antisymmetry, and some other values in two blocks are likewise suppressed. The values ν_{0i} are the canonical multipliers found in I.

These formulae may be re-expressed by using any weak second-class equivalent formulae. This is because the second-class constraints have vanishing DBs with the canonical variables and their functions, and are identities in the DB formalism (the symbol '$\approx$' is replaced by '$=$' in this formalism, the second-class constraints becoming strong equations). Thus certain rows and columns of the table could be inferred from other rows and columns, e.g. the columns for λ_{0j} and m_0c^2 could be inferred from those for p^μ and u_j^μ using

$$\lambda_{0j} = p^\alpha u_{j\alpha}, \qquad m_0^2c^4 = p^\alpha p_\alpha + \lambda_{0j}\lambda_{0j} \tag{3.10}$$

(I, equation (A2.1)). We have been unable to establish the mutual DBs of the u's and π's, so we retain these columns. In contrast, the columns for $c^{-2}\Pi_0$ and Π_{0j} are

† Non-zero values in the blocks have been analysed by choosing the 12 parameters arising in the matrices $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$ in a special way. By making special choices (e.g. $\hat{\boldsymbol{v}} = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$) it is possible to identify values in the blocks that are proportional either to four-vectors or to Lorentz-invariant triples. The 0's and 1's in the table have been established by repeated use of the random number generator to assign the 12 parameters. The algebraic expressions given in table 1 have all been verified in subsequent runs.

quite unnecessary since it is known already that their DBs with all the canonical variables vanish, since they themselves vanish, being second-class constraint functions. The non-zero values in these columns confirm this.

The numerical results indicate that the mutual DBs of unequal u's do not vanish. (Corben's simple assumptions of vanishing brackets for these variables (1968, p 255) appear not to apply.) None of the DBs of the u's with the λ's vanishes, but some of the mutual DBs of the π's do.

3.4. PBs and DBs of position, momentum and the special combinations

Because the 36 elements of the first block of table 1 all vanish, the DBs of these variables are the same as their corresponding PBs. In table 2 below, we give this list of PBs (and hence DBs). Note that no algebraic calculations for DBs have been performed other than those for the x's and the p's.

Table 2. DBs of x^α, p^α and the special combinations (3.3).

	$x^\mu\}^*$	$p^\mu\}^*$	$m^{\mu\nu}\}^*$	$s^{\mu\nu}\}^*$	$cs_j\}^*$	$j^{\mu\nu}\}^*$
$\{x^\alpha,$	0	$g^{\alpha\mu}$	$g^{\alpha\mu}x^\nu - g^{\alpha\nu}x^\mu$	0	0	$\{x^\alpha, m^{\mu\nu}\}^*$
$\{p^\alpha,$		0	$g^{\alpha\mu}p^\nu - g^{\alpha\nu}p^\mu$	0	0	$\{p^\alpha, m^{\mu\nu}\}^*$
$\{m^{\alpha\beta},$			$g^{\alpha\nu}m^{\beta\mu} + g^{\beta\mu}m^{\alpha\nu} - g^{\alpha\mu}m^{\beta\nu} - g^{\beta\nu}m^{\alpha\mu}$	0	0	$\{m^{\alpha\beta}, m^{\mu\nu}\}^*$
$\{s^{\alpha\beta},$				$g^{\alpha\nu}s^{\beta\mu} + g^{\beta\mu}s^{\alpha\nu} - g^{\alpha\mu}s^{\beta\nu} - g^{\beta\nu}s^{\alpha\mu}$	0	$\{s^{\alpha\beta}, s^{\mu\nu}\}^*$
$\{cs_i,$					$\varepsilon_{ijk}cs_k$	0
$\{j^{\alpha\beta},$						$\{m^{\alpha\beta}, m^{\mu\nu}\}^* + \{s^{\alpha\beta}, s^{\mu\nu}\}^*$

The canonical p^α and $j^{\alpha\beta}$ generate the translations and the Lorentz rotations of the 10-parameter Poincaré group and it is not surprising that the DB relations for these variables are identical to the basic Lie bracket relations for the Poincaré group:

$$\begin{aligned} &\{p^\alpha, p^\mu\}^* = 0, \qquad \{p^\alpha, j^{\mu\nu}\}^* = g^{\alpha\mu}p^\nu - g^{\alpha\nu}p^\mu, \\ &\{j^{\alpha\beta}, j^{\mu\nu}\}^* = g^{\alpha\nu}j^{\beta\mu} + g^{\beta\mu}j^{\alpha\nu} - g^{\alpha\mu}j^{\beta\nu} - g^{\beta\nu}j^{\alpha\mu}. \end{aligned} \tag{3.11}$$

However, the vanishing of the mutual DBs of the x's does not also follow from simple assumptions (cf the treatment of Hanson and Regge (1974) which, though leading to (3.11), does not lead to vanishing mutual DBs of the x's†).

The generators in (3.11) may be realised by differential operators, the bracket relations (3.11) being interpreted as commutation relations ($\{\ \}^* \to [\ ,\ ]$). For table 2

† Because a different condition of spin is used which effectively allows only straight-line motion in the field-free case, Hanson and Regge's treatment does not lead to vanishing DBs of the x's. Transformation of the x's to Pryce–Newton–Wigner variables (Pryce 1948, Newton and Wigner 1949) avoids substantial quantum ordering problems and leads to the vanishing of the brackets for the space parts of the new variables only. In contrast to their treatment, none of the DBs given here requires the transformation and all are fully covariant. The six operators of (3.12) for $s^{\alpha\beta}$ appear to generalise three non-covariant operators found by Hanson and Regge (1974, p 536, formula (3.73)) after the transformation to Pryce–Newton–Wigner variables. The covariant operators (3.12) demonstrate how spin may be included in the differential representation of an asymmetric particle, generalising the representation for x's and p's.

we have

$$1\to 1,\qquad x^\alpha\to x^\alpha,\qquad p^\alpha\to -\partial/\partial x_\alpha,\qquad s^{\alpha\beta}\to -(u_i^\beta\partial/\partial u_{i\alpha}-u_i^\alpha\partial/\partial u_{i\beta}),$$
$$cs_i\to -\varepsilon_{ijk}u_j^\sigma\partial/\partial u_k^\sigma,\qquad m^{\alpha\beta}\to -(x^\beta\partial/\partial x_\alpha-x^\alpha\partial/\partial x_\beta). \tag{3.12}$$

These are connected by constraint equations. Thus the Poincaré group is insufficient for the complete canonical description of the free particle equations. The canonical theory of DBs is an essential process for constructing the classical formalism.

3.5. Reduction of the system to a sufficient set of variables

The second-class constraints allow us to find values of many DBs from others, and we may attempt to reduce the system to a sufficient set of variables. A certain reduction has already been achieved in deducing the algorithm that allots values to the canonical variables from $2n_1=16$ parameters†. However, it is feasible to try to find a reduced system that has some of the properties of a Lie algebra. Table 2 is not quite sufficient for the purposes of representing all the canonical variables using the second-class constraints. Below we give the number of independent functions already existing in the table and attempt to find the independent functions which form the maximum number for the canonical system.

The number of independent functions of table 2 is 15‡. These functions must be supplemented by $2n_1-15=1$ further independent variable, and it is clear that this variable must arise from the λ's since these are not completely determined from the variables of table 2 by using the constraints. There is no convenient single variable that may be adjoined to the table in such a way that this observable forms, with the others, a Lie algebra. The best we can achieve is to assign all three λ's subject to two constraint equations (this would require knowledge of the complete table of DBs). Without the λ's, 17 functions x^α, p^α, $s^{\alpha\beta}$, cs_i, when supplemented by the element 1, form a Lie subalgebra in which there exist two constraint equations—conveniently, equations connecting the Casimir invariants: $\frac{1}{2}s^{\alpha\beta}s_{\alpha\beta}=c^2s_is_i$, $s^{\alpha\beta}s^*_{\alpha\beta}=0$. These reduce the system to 15 independent functions (and the unit element). We have justified this fact in the footnote by listing the constraints that involve these variables.

4. Conclusion

The canonical theory presented here is somewhat different from those that seek to construct (rather than to derive) canonical formalisms from a symmetry group (e.g. the so-called 'canonical realisations' of the Galilean, Poincaré, and other groups: see e.g. Pauri and Prosperi (1975) and the method of constructing a canonical formalism

† The six first-class functions Π_{ij} and their six conjugates λ_{ij} correctly increase the number of observables for this representation from 16 to 28, the model having 14 degrees of freedom.

‡ The number of independent functions already existing in the table is obtained thus. The three functions s_i may be considered to be adjoined to the others of table 2 in such a way that $c^2s_is_i=\frac{1}{2}s^{\alpha\beta}s_{\alpha\beta}$ so that we have effectively only two functions adjoined to the other functions p^α, $s^{\alpha\beta}$ of table 2, in terms of which the only other condition imposed by the second-class constraints is the single condition $s^{\alpha\beta}s^*_{\alpha\beta}=0$, effectively reducing the six functions $s^{\alpha\beta}$ to five. (This condition arises essentially from the Frenkel condition of spin as a result of an identity which already holds for *any* antisymmetric tensor: $\frac{1}{2}s^{\alpha\beta}s_{\alpha\beta}s_{\mu\nu}+s_\mu{}^\alpha s_{\alpha\beta}s^\beta{}_\nu\equiv\frac{1}{4}s^{\alpha\beta}s^*_{\alpha\beta}s^*_{\mu\nu}$.) These two conditions are the only conditions imposed by the constraints on the variables of table 2, and the independent functions of the table number 15: x^α (four functions), p^α (four functions), $s^{\alpha\beta}$ (five functions), cs_i (two functions).

from a symmetry group summarised by Mann (1974)). In these theories (i) the PB is almost universally accepted as agreeing with the Lie bracket and (ii) the contact between the physical system and the abstract algebraic formalism is made essentially by postulate (or even by guesswork), e.g. for the energy, momentum and angular momentum observables.

In the canonical formalism given here, we have derived the algebraic structure from the physical, and it is evident that the elements of Poincaré algebra are insufficient for a complete canonical description for the motion of a free particle with spin. Our formalism is, therefore, unlike those constructions that include spin in the realisations of the Poincaré group. Our formalism requires the use of constraint equations, which are not found in conventional theories.

We can give some comparisons between this work and the postulated formalism of Corben (1968). The previous theory lacked the means for accurate derivation of the brackets and certain inconsistencies developed in these treatments, which were based largely on postulation of certain fundamental brackets. The present treatment appears to provide the only means for deriving the bracket relations accurately for the model.

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